Lectures 9

Differential Equations

- Solutions about regular ordinary points
- Solutions about regular singular pts
- Frobenius Method

Differential Equations:

Let

$$u'' + p(z)u' + q(z)u = 0 \qquad (1)$$

- $p(z)$ and $q(z)$ are analytic in a certain region R, except perhaps some finite number of isolated singularities.

- If $p(z)$ and $q(z)$ are analytic at a point $z = z_0$ of R is called an Ordinary point of the differential equation

- We shall show that there exist two linearly independent solutions of (1) in the neighborhood of ordinary points. These solution analytic in the neighborhood of $z = z_0$.

- If z_0 is an isolated singular point of $p(z)$ or $q(z)$ then such a point is called the singular point of (1). Two types of singular points

(i) Regular singular points: $z = z_0$ is a regular singular point if

$$(z-z_0)\,p(z) = \text{analytic in the neighborhood of } z_0$$
$$(z-z_0)^2 q(z) = \text{analytic}$$

(ii) Irregular singular points: If not a regular singular point it is called an irregular singular point.

Frobenius Method: In the neighborhood of a regular singular point the solution

$$u_1(z) = (z-z_0)^{r_1} \sum_{n=0}^{\infty} c_n^1 (z-z_0)^n$$

$$u_2(z) = (z-z_0)^{r_2} \sum c_n^2 (z-z_0)^n \qquad r_1 \neq r_2$$

or

$$u_1(z) = (z-z_0)^{r_1} \sum c_n^1 (z-z_0)^n$$

$$u_2(z) = C\, u_1(z) \ln(z-z_0) + (z-z_0)^{r_2} \sum c_n^2 (z-z_0)^n$$

- If z_0 is an irregular point FM doesn't work. It need an Laurent expansions.

Existence and uniqueness: about a regular singular pt.

Let $u'' + p(z)u' + q(z)u = 0 \qquad z \in \mathbb{R}$
with
$u(z_0) = a, \quad u'(z_0) = b$

Define: $u = f\, e^{-\frac{1}{2}\int_{z_0}^{z} p(\xi)d\xi}$

$$u' = \left(f' - \tfrac{1}{2} f p\right) e^{-1/2 \int}$$

$$u'' = \left(f'' - f'p - \tfrac{1}{2} f p' + \tfrac{1}{4} f p^2\right)$$

$$u'' + pu' + qu = f'' - \cancel{f'p} - \tfrac{1}{2} f p' + \tfrac{1}{4} f p^2 + \cancel{pf'} - \tfrac{1}{2} f p^2 + qf = 0$$

$$f'' + h(z) f = 0, \qquad h(z) = q - \tfrac{1}{2} p' - \tfrac{1}{4} p^2$$

Define a sequence of functions

$f_0(z), f_1(z), \dots, f_k(z), \dots$

Defined by

$$f_n''(z) + k(z) f_{n-1} = 0 \qquad n = 1, 2, \dots,$$

Lemma

If the summation $\sum_{n=0}^{\infty} f_n$ is uniformly convergent it solves the DE

f_0 is arbitrary

$f_0'' = 0$

$$f(z) = \sum_{n=1}^{\infty} f_n$$

$$f'' = \left(\sum f_n\right)'' = \sum f_n'' = -\sum k f_{n-1}$$

$$= -kf \ni f'' + kf = 0 \quad \text{"solves DE"}$$

Lemma:

$$f_n(z) = \int_{z_0}^{z} (z' - z)\, k(z') f_{n-1}(z')\, dz' \quad , n = 1, 2, \dots$$

proof:

$$f_n'(z) = -\int_{z_0}^{z} k(z') f_{n-1}(z')\, dz'$$

$f_n'' = -k(z) f_{n-1}(z)$ ✓

$$f_n(z) = -\int_{z_0}^{z} dz' \int_{z_0}^{z'} k(z'') f_{n-1}(z'')\, dz''$$

$$= -\left(z' \int_{z_0}^{z'} k(z'') f_{n-1}(z'')\, dz''\right)\Bigg|_{z_0}^{z}$$

$$+ \int_{z_0}^{z} z' k(z') f_{n-1}(z')\, dz'$$

$$f_n(z) = \int_{z_0}^{z} \underset{(z'-z)}{\cancel{(z'-z_0)}}\, k(z') f_{n-1}(z')\, dz'$$

(DE 7)

$$f_n(z) = \int_{z_0}^{z} (z'-z_0)\, k(z')\, f_{n-1}(z')\, dz'$$

Lemma: $$|f_m(z)| \le |\max f_0|\, |\max k(z)|^m \frac{|z-z_0|^{2m}}{m!}$$

true for $m=0$, assume for $n-1$. Take

$$|f_n(z)| \le \int_{z_0}^{z} |z'-z_0|\, |k(z')| \left(|\max f_0|\, |\max k|^{n-1} |z'-z_0|^{2n-2} \frac{1}{(n-1)!} dz' \right)$$

$$|f_n(z)| \le |\max f_0|\, |\max k|^n \frac{1}{(n-1)!} \int_{z_0}^{z} |z'-z_0|\, |z'-z_0|^{2n-2} dz'$$

$0 < t \le 1$

$z' = z_0 + t(z-z_0)$

$$\le |\max f_0|\, |\max k|^n \frac{1}{(n-1)!} |z-z_0|^{2n} \int_0^1 t\, (1-t)^{\cancel{2n-2}}\, dt$$

$$\le |\max f_0|\, |\max k|^n \frac{1}{n!} |z-z_0|^{2n}$$

Then

$$|\Sigma f_n| \le \Sigma |f_n| = |\max f_0|\, e^{|\max k|\, |z-z_0|^2}$$

$$|\Sigma f_n| \le |\max f_0|\, e^{|\max f_0|\, |z-z_0|^2}$$

uniformly convergent.

hence $f = \Sigma f_n$ solves (1). $f_0'' = 0$

$f(z_0) = a$, $f'(z_0) = b$ | $f_0 = a + (z-z_0) b$

$u = f e^{-\frac{1}{2}\int^z}$

$u(z_0) = f(z_0) = a$

$u'(z_0) = f'(z_0) - \frac{1}{2} a\, p(z_0) = b$

DE 4

Solution is unique

Let $w = f - \tilde{f}$ $\quad \tilde{f}$ solves the same DE with same BCs

$w'' + h(z)w = 0 \quad w(z_0) = 0, \quad w'(z_0) = 0$

$w''(z_0) = 0, \ \ldots, \ w^{(k)}(z_0) = 0$

zero solution $w = 0$ unique.

An application: Example:

$u'' - 2zu' + 2\lambda u = 0$

$z = 0$ is an ordinary point. The only singular pt is at $z = \infty$

$u(z) = \sum c_n z^n$

$\sum [(n+2)(n+1)c_{n+2} - 2nc_n + 2\lambda c_n] z^n = 0$

$$c_{n+2} = \frac{2(n-\lambda)}{(n+2)(n+1)} c_n$$

starting with c_0 we can generate

$u_1(z) = \sum c_{2n} z^{2n}$

starting with c_1 we can generate

$u_2(z) = \sum c_{2n+1} z^{2n+1}$

If λ is an integer: If $\lambda = 2N$ $\quad u_1$ is a polynom
$\lambda = 2N+1 \quad u_2$ is a polynom

These are the Hermite polyn.

DE5

Example: $u'' - 2zu' + 2\lambda u = 0$

$$(L + 2\lambda)u = 0 \qquad L = \frac{d^2}{dz^2} - 2z\frac{d}{dz}$$

$$= z^3\frac{d}{dz}\left(\frac{1}{z^2}\frac{d}{dz}\right.$$

$z=0$ is a regular point: $u(z) = \sum_{n=0}^{\infty} c_n z^n$

$$\sum_{n=2} n(n-1)c_n z^{n-2} - 2\sum n z^n + 2\lambda \sum c_n z^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)c_{n+2} + (-2n+2\lambda)c_n\right]z^n = 0$$

$$(n+2)(n+1)\,c_{n+2} = -2(\lambda-n)\,c_n \qquad n=0,1,2,\dots$$

$$c_{n+2} = \frac{2(n-\lambda)}{(n+2)(n+1)}\,c_n \qquad n=0,1,2$$

c_0, c_1 are arbitrary ; Even solutions

$$c_2 = \frac{-2\lambda}{2}c_0 = -\lambda c_0$$

$$c_4 = \frac{2(2-\lambda)}{4\cdot 3}c_2 = -\lambda\,\frac{2(2-\lambda)}{4\cdot 3}c_0$$

$$c_6 = \frac{2(4-\lambda)}{6\,5}c_4 = -\lambda\,\frac{2(2-\lambda)}{4\cdot 3}\cdot\frac{2(4-\lambda)}{6\cdot 5}$$

$$c_{2n} = -\lambda\,\frac{2^n\,(2n-2-\lambda)(2n-4-\lambda)\cdots(2-\lambda)}{(2n)!}c_0$$

$$u_1(z) = \sum_{n=0}^{\infty} c_{2n} z^{2n}$$

odd solution $\Rightarrow$

$$c_3 = \frac{2(1-\lambda)}{2\cdot 3} c_1$$

$$c_5 = \frac{2(3-\lambda)}{5\cdot 4} \, \frac{2(1-\lambda)}{2\cdot 3} c_1$$

$$c_{2n+1} = \frac{2^n (2n-1-\lambda)(2n-3-\lambda)\cdots(1-\lambda)}{(2n+1)!} c_1$$

$$u_2(t) = \sum_{n=0}^{\infty} c_{2n+1} t^{2n+1}$$

The general soln. $u = u_1 + u_2$.

If λ is an integer:

λ = even integer $u_1(t)$ becomes a polynomial
λ = odd integer $u_2(t)$ - - -

These are Hermite polynomials we consider in the section "Orthogonal Polynomials"

Solution of a DE in a neighborhood of a Regular singular point.

$$L_z u = \frac{d^2u}{dz^2} + p(z)\frac{du}{dz} + q(z)u = 0 \qquad (1)$$

We assume that all points of D are ordinary points of Eq. (1) except $z = z_0$ which is a regular singular point of (1). This means that the functions

$$A(z) = (z-z_0)\,p(z) \quad , \quad B(z) = (z-z_0)^2 q(z) \qquad (2)$$

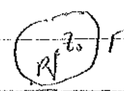
are analytic everywhere in D including the point z_0, hence

$$A(z) = \sum_{n=0} a_n (z-z_0)^n \quad , \quad B(z) = \sum b_n (z-z_0)^n \qquad (3)$$

where from the Cauchy theorem:

$$a_n = \frac{1}{2\pi i}\int_\Gamma \frac{A(z')\,dz'}{(z'-z_0)^{n+1}} \quad , \quad b_n = \frac{1}{2\pi i}\int_\Gamma \frac{B(z')\,dz'}{(z'-z_0)^{n+1}} \qquad (4)$$

where



Γ is a circle of radius R surrounding z_0

Then we try to find a solution of (1) in the form of a series

$$u(z) = (z-z_0)^r \sum C_n (z-z_0)^n \qquad (5)$$

where r and C_n $(n = 0, 1, \dots)$ are constants to be determined. Using (5) and (3) in (1) we get

(DE8)

$$\sum c_n (n+r)(n+r-1) \cancel{c_n} (z-z_0)^{n+r-2}$$
$$+ \sum a_n (z-z_0)^{n-1} \sum c_n (n+r)(z-z_0)^{n+r-1}$$
$$+ \sum b_n (z-z_0)^{n-2} \sum c_n (z-z_0)^{n+r} = 0$$

$$\sum c_n (n+r)(n+r-1) (z-z_0)^n + \sum a_m (z-z_0)^m \sum c_n (n+r)(z-z_0)^n$$
$$+ \sum b_m (z-z_0)^m \sum c_n (z-z_0)^n = 0$$

use $\sum p_k z^k \sum q_l z^l = \sum s_m z^m$

$$s_m = \sum_{i=0}^{m} p_{m-i} q_i$$

$$\Rightarrow \sum \left[c_n (n+r)(n+r-1) + \sum_{m=0}^{n} \left[c_m a_{n-m} (m+r) + c_m b_{n-m} \right] \right] (z-z_0)^n = 0 \quad (6)$$

here combining c_n's

$$c_n \lambda_0 (n+r) + \sum_{m=0}^{n-1} c_m \left[(m+r) a_{n-m} + b_{n-m} \right] = 0 \quad (7)$$

(recursion relations)

where $\lambda_0 (n+r) \equiv (n+r)(n+r-1) + a_0 (n+r) + b_0$ (8)

To solve Eq. (1) we have to determine all c_n's by using (7). For $n=0$ we have

$n=0$ $\quad c_0 \lambda_0(r) = 0 \qquad c_0 \neq 0$ arbitrary (9)

$\lambda_0(r) = 0$ "the indicial equation"

$$r^2 + (a_0 - 1) r + b_0 = 0$$

$$r_1 + r_2 = 1 - a_0 \quad , \quad r_1 r_2 = b_0$$

The roots of the indicial equation are important in the solution of (1). We assume

$$Re(r_1) \geq Re(r_2) \quad (10)$$

$r(r-1) + a_0 r + b_0 \Rightarrow$

also

$r_1 + r_2 = 1 - a_0$

$$Re(r_1) \geq \frac{1}{2}(1-a_0) \quad (11)$$

$Re(r_1) \geq \frac{1}{2}(1-a_0)$

$$\lambda_0(r) = r^2 + (a_0-1)r + b_0 \quad (12)$$

Lemma 1. There always exists one solution of Eq. (1) in the form of series expansion

$$u(z) = (z-z_0)^r \sum_{n=1}^{\infty} c_n (z-z_0)^n$$

with $r = r_1$

Lemma 2. With $r = r_2$ Eq. (1) yields a second solution (linearly independent) provided the roots r_1 and r_2 do not differ by an integer.

Proof of (1): To prove (1) we assume that $\lambda_0(n+r)$ never vanishes for $n \geq 0$.

λ_0 can vanish only if $r_1 + n = r_2$ but because of inequality $Re(r_1) \geq Re(r_2)$ this cannot happen. $r_1 - r_2 \neq N$

Since they are analytic in D $A(z)$ and $B(z)$ are ~~analytic~~ bounded in D hence let

$$|A(z)| < \Lambda_1 \quad \text{and} \quad |B(z)| < \Lambda_2 \quad \text{for } z \in D$$

$$\Rightarrow |a_n| \le \frac{\Lambda_1}{R^n}, \quad |b_n| \le \frac{\Lambda_2}{R^n}$$

$\Rightarrow$ (1) $\lambda_0(r_1+n) = (r_1+n)(r_1+n-1) + a_0(n+r_1) + b_0$

$= r_1(r_1-1) + a_0 r_1 + b_0$

$+ n(r_1+n-1) + a_0 n + r_1 n$

r_1+n-1

$= n^2 + 2nr_1 + a_0 n - n$

$= n(n+2r_1-1+a_0)$

$= n^2 + n\underbrace{(2r_1-1+a_0)}_{\geq 0}$

$\Rightarrow \lambda_0(r_1+n) \geq n^2 \qquad \frac{1}{\lambda_0} \le \frac{1}{n^2}$

$$C_n = -\sum_{m=1}^{n-1} C_m \frac{(m+r_1)a_{n-m} + b_{n-m}}{\lambda_0(r_1+n)}$$

$$|C_n| \le \sum_{m=1}^{n-1} \frac{1}{n^2}|C_m| \frac{(m+|r_1|)\Lambda_1 + \Lambda_2}{R^{n-m}}$$

$$\frac{(m+|r_1|)\Lambda_1 + \Lambda_2}{n} < \underbrace{|r_1| + \Lambda_1 + \Lambda_2}_{\text{fixed number}} \le K \quad \text{indep of } m$$

we done this coz as >1

$$|C_n| \le \sum \frac{K}{n} \frac{|C_m|}{R^{n-m}}$$

hence we get

$$|c_n| \le \frac{K}{n} \sum_{m=0}^{n-1} \frac{|c_m|}{R^{n-m}} \quad , \quad K = |r_1| + A_1 + A_2 > 1$$

one can show that

$$|c_n| \le \left(\frac{K}{R}\right)^n |c_0|$$

proof:

$$|c_1| \le \frac{K}{R} |c_0|$$

$$|c_2| \le \frac{K}{2}\left(\frac{|c_0|}{R^2} + \frac{|c_1|}{R}\right) < \frac{K}{2}\left(\frac{1}{R^2} + \frac{K}{R^2}\right)|c_0|$$

$$\le \frac{K^2}{R^2}|c_0|$$

$$\Rightarrow \quad u(z) = (z-z_0)^r \sum c_n (z-z_0)^n$$

converges

$$|u(z)| \le (z-z_0)^r \sum \left(\frac{K}{R}\right)^n (z-z_0)^n$$

converges in D with the radius of convergence

$$\rho = \frac{R}{K} < R$$

$$\frac{1}{1 - \frac{K}{R}(z-z_0)} \quad \Rightarrow \quad \left|\frac{K}{R}|z-z_0|\right| \le 1$$

$$|z-z_0| \le \frac{R}{K}$$

$r_1 - r_2 \notin N$ (DE12)

Example: $2z^2y'' - zy' + (1+z)y = 0$

$r_1 = 1, \quad r_2 = 1/2 \qquad y = \sum_{n=0} a_n (z)^{n+r}$

$y' = \sum a_n (r+n) z^{r+n-1}$

$y'' = \sum a_n (r+n)(r+n-1) z^{r+n-2}$

$\sum 2a_n (r+n)(r+n-1) z^n - \sum a_n (r+n) z^n$

$+ \sum a_n z^n + \sum a_n z^{n+1} = 0$

$2a_0 r(r-1) - a_0 r + a_0$

$+ \sum_{n=1} a_n [2(r+n)(r+n-1) - (r+n) + 1] z^n$

$+ \sum_0 a_n z^{n+1} = 0$

$a_{n-1} z^n \qquad n+1 = m$

$a_n [2(r+n)(r+n-1) - (r+n) + 1] + a_{n-1} = 0 \quad n \geq 1$

$2r(r-1) - r + 1 = 2r^2 - 3r + 1 = (2r-1)(r-1)$

$\boxed{r_1 = 1, \; r_2 = 1/2,}$

$a_n = \dfrac{-a_{n-1}}{(r+n-1)[2(r+n)-1]} \qquad n \geq 1$

a_0 free

DE 13

$r_1 = 1, \quad a_n = \dfrac{-a_{n-1}}{n(2n+1)} \quad n \geq 1$

$$a_n = \frac{(-1)^n a_0}{[3 \cdot 5 \cdot 7 \cdots (2n+1)]\, n!} \quad n \geq 1$$

$$u_1(z) = z\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{[3 \cdot 5 \cdot 7 \cdots (2n+1)]\, n!}\right]$$

ratio test: $\lim_{n\to\infty} \left|\dfrac{u_{n+1}}{u_n}\right| = 0$ series converges for all z

$r = r_2 = 1/2 \quad a_n = -\dfrac{a_{n-1}}{n(2n-1)} \quad n \geq 1$

$$a_n = \frac{(-1)^n}{n!\,(1 \cdot 3 \cdot 5 \cdots (2n-1))} \quad n \geq 1$$

$$u_2(z) = z^{1/2}\left[1 + \sum_{n=1} \frac{(-1)^n z^n}{n!\,[1 \cdot 3 \cdot 5 \cdots (2n-1)]}\right]$$

general soln.

$u = c_1 y_1(z) + c_2 y_2(z)$

Frobenius method

DE 14

Near a regular singular point we have the following cases

I. $r_1 - r_2 =$ Not an integer
follow the standard procedure

II. $r_1 - r_2 =$ Integer $\neq 0$

a) Recursion relation is consistent.

b) Recursion relation is not consistent.

$r_1 \rightarrow u_1(z)$

$$u_2(z) = c\,u_1(z)\ln z + (z-z_0)^{r_2}\left(1 + \sum c_n(r_2)(z-z_0)^n\right)$$

$$= c\,u_1(z)\ln z + (z-z_0)^{r_2}\left[1 + \sum c_n(r_2)(z-z_0)^n\right]$$

$$c_n(r_2) = \frac{d}{dr}\left[(r-r_2)c_n(r)\right]\Big|_{r=r_2} \qquad n = 1, 2, \dots$$

III. $r_1 = r_2$

$r_1 \leftrightarrow u_1(z)$

$$u_2(z) = c\,u_1(z)\ln z + (z-z_0)^{r_1}\sum_{n=1} b_n'(r_1)(z-z_0)^n$$

$$b_n'(r_1) = \frac{d\,c_n(r)}{dr}\Big|_{r=r_1}$$

Examples

(1)

$$2z(1+z)u'' + (3+z)u' - zu = 0$$

- near the regular singular point $z=-1$.

$$p(z) = \frac{3+z}{2z(1+z)}$$

$$\lim_{z\to -1} (1+z)p(z) = -1 = a_0$$

$$q(z) = -\frac{z}{2z(1+z)}$$

$$\lim_{z\to -1} (1+z)^2 q(z) = -0 = b_0 = 0$$

indical equation

$$r(r-1) + a_0 r + b_0 = 0$$

$$r(r-1) - r = 0 \qquad r(r-2) = 0$$

$$\boxed{r_1 = 2, \quad r_2 = 0}$$

- near the regular singular point $z=0$

$$\lim_{z\to 0} z\,p(z) = 3/2 = a_0$$

$$\lim_{z\to 0} z^2 q(z) = 0 = b_0$$

$$r(r-1) + 3/2\, r = 0 \qquad \boxed{r_1 = 0, \ r_2 = -1/2}$$

DE 1$\frac{1}{4}$

(2) $r = r_1 = 0$

$$\sum_{n=0} n(n-3) c_n z^{n-1} + \sum_{n=0} (2 - n^2 + n) c_n z^n = 0$$

$(n+1)(n-2) c_{n+1} - (n+1)(n-2)$

$c_n = 0$

$$-2c_1 - 2c_2 z + 0 + c_4 z^3 + 10 c_5 z^4$$ $(n+1)(n-2)(c_{n+1} - c_n) = 0$

$$+ 2c_0 + 2c_1 z + 0 + 4c_3 z^3 + 10 c_4 z^4 + \cdots$$

let $c_3 = 0 \Rightarrow c_4 = 0$

$c_1 = c_0, \quad c_2 = c_1 = c_0$

$u_2(t) = 1 + z + z^2$

$$u(z) = a(1 + z + z^2) + \frac{b z^3}{1 - z}$$

$n = 1 + m$

$$\sum_{-1} (m+1)(m-2) c_{m+1} z^n$$

$(m+1)(m-2) c_{m+1} + (2 - n)(1 + n)$

$c_{m+1} - c_m = 0$ $(n-2)(m+1)[c_{n+1} - c_n]$

$m = 0, 1, 2$

$c_1 = c_0$

$c_2 = c_1 = c_0$

c_0 which

$c_1 = c_0$

$c_2 = c_1$

$n = 2 \quad 0 = 0$

$c_3 = c_4 = 0,$

DE 16

Example:

1) $z(1-z)u'' - 2u' + 2u = 0$

$$p(z) = -\frac{2}{z(1-z)} = -2\left(\frac{1}{z} + \frac{1}{1-z}\right)$$

$$z\,p(z) = -2\left(1 + \frac{z}{1-z}\right) \qquad a_0 = (-2)$$

$$q(z) = \frac{2}{z(1-z)} = 2\left(\frac{1}{z} + \frac{1}{1-z}\right), \quad b_0 = 2$$

$$\lambda_0(r) = r^2 - r - 2r = r(r-3)$$

$$r_1 = 3, \quad r_2 = 0$$

1) $r_1 = 3$ $\qquad u(z) = (z-z_0)^r \sum c_n (z-z_0)^n$

$$u(z) = \sum c_n z^{n+r}$$

$$z(1-z)\sum (n+r)(n+r-1)c_n z^{n+r-2} - 2\sum (n+r)c_n z^{n+r-1}$$

$$+2\sum c_n z^{n+r} = 0$$

$$\sum \left[(n+r)(n+r-1) - 2(n+r)\right]c_n z^{n+r-1} + \sum c_n\left[2-(n+r)(n+r-1)\right] z^{n+r} = 0$$

$$\sum \left[(n+r)(n+r-3)c_n z^{n-1} + \sum (2-(n+r)(n+r-1))c_n z^n = 0\right.$$

$$r(r-3)c_0 z^{-1} + \sum_{n=1} (n+r)(n+r-3)c_n z^{n-1} + \dots$$

$$n-1 = m \qquad r = 3$$

$$\sum \left[c_{n+1}(n+4)(n+1) + (2-(n+3)(n+2))c_n\right] z^n = 0$$

$$c_{n+1}(n+4)(n+1) = (n^2+5n+4)c_n = (n+1)(n+4)c_n$$

$$c_{n+1} = c_n \quad \text{for all } n \geq 1.$$

$$u(x) = c_0 z^3(1 + x + x^2 + \dots) = \frac{z^3 c_0}{1-x}$$

Lemma 1: Let $z = z_0$ be a regular singular point of $u'' + p(z)u' + q(z)u = 0$ with $r_1 - r_2 = N$ a positive integer then

$$u_2(z) = C\, u_1(z) \log(z-z_0) + (z-z_0)^{r_2} \sum c_n (z-z_0)^n$$

where C is a const and $u_1(z)$ corresponds to the solution with $r = r_1$

proof: let $u_1(z)$ be the solution (1) with $r = r_1$. Then

$$u_2 = u_1 \int \frac{dz'}{u_1^2(z')} e^{-\int p(z')dz'}$$

$$p(z) = \frac{a_0}{(z-z_0)} + a_1 + a_2 (z-z_0) + \cdots$$

$$\int p(z')dz' = a_0 \log(z-z_0) + f(z)$$

$$e^{-\int} = (z-z_0)^{-a_0} e^{-f(z)}$$

$$u_1(z) = (z-z_0)^{r_1} g(z)$$

$$\int \frac{dz'}{u_1^2} e^{-\int} = \int \frac{dz'}{(z'-z_0)^{2r_1+a_0}} e^{-f(z)}/g(z)$$

$$= \int \frac{dz'}{(z'-z_0)^{2r_1+a_0}} H(z').$$

i) $r_1 - r_2 = N$

ii) $r_1 + r_2 = 1 - a_0$

$2r_1 = N + 1 - a_0$

$$= \int \frac{dz'}{(z'-z_0)^{1+N}} H(z)$$

$$= \int \frac{dz'}{(z-z_0)^{1+N}} \left[d_0 + d_1 (z-z_0) + \dots + d_N (z-z_0)^N + \dots \right]$$

$$= \int \left(\frac{d_N}{z-z_0} dz' + \dots \right)$$

$$= d_N \ln(z-z_0) + G(z) \quad , \quad G = \frac{d_0}{(z-z_0)^N} + \dots$$

$$\Rightarrow u_2(z) = C\, u_1(z) \ln(z-z_0) + u_1 G(z)$$

$$u_1 G = u_1 \left[\frac{a_1}{(z-z_0)^N} + \dots \right]$$

$$= (z-z_0)^{-N} u_1 \left[a_1 + \dots \right]$$

analtic

$$= (z-z_0)^{r_1 - N} \text{ Analli}$$

$$= (z-z_0)^{r_2} \text{ anally}$$

$$u_2(z) = C u_1(z) \ln(z-z_0) + (z-z_0)^{r_2} \sum_{n=1}^{\infty} c_n (z-z_0)^n$$

1. Frobenius Method

Lemma 1 2. Proof of log singularity when $r_1 - r_2 = N, 0$

3. Proof of FM

$$u(z,r) = (z-z_0)^r \sum_{n=0}^{\infty} c_n(r)(z-z_0)^n$$

$$L(u) = \lambda_0(r)(z-z_0)^{r-2} c_0$$

$$\lambda_0(n+r)\, c_n(r) + \sum_{m=0}^{n-1} c_m(r)\left[(m+r)a_{n-m} + b_{n-m}\right] = 0$$

For $r = r_1$ $\quad u(z,r) \to u_1(z)$

$$u_1(z) = (z-z_0)^{r_1} \sum_{n=0}^{\infty} c_n(r_1)(z-z_0)^n$$

$r = r_2$ $\quad (r_1 - r_2 \neq N, 0)$.

$$u_2(z) = (z-z_0)^{r_2} \sum c_n(r_2)(z-z_0)^n$$

$\therefore$ $c_n(r_1)$ and $c_n(r_2)$ are found from the recursion relations.

When $r_1 - r_2 = N, 0$

for $r = r_1$ $\quad u(z,r) = u_1(z)$

$$u_1(z) = (z-z_0)^{r_1} \sum c_n(r_1)(z-z_0)^n$$

where $c_n(r_1)$ is found from the recursion relt.

for $r = r_2$ the recursion relation $\lambda_0(N+r_2)$
$= \lambda(r_1) = 0$ hence there exists either
an inconsistency $C_n(r_2)$ doesn't exist
or the recursion relation are consistent.

a) If the recursion relation are ~~not~~ consistent find $C_N(r_2)$ from the r.r.

b) If the recursion relation are NOT consistent $C_n(r_2)$ doesn't exist, hence the standard form for $u(r,t)$ doesn't work

For this case we can find $u_2(z)$ in the following way. since

$$L(u) = \lambda_0(r)(z-z_0)^{r-2} C_0$$

Lemma 2. ~~1.~~ $(r-r_2)\,u(r,t) \xrightarrow{\text{as } r\to r_2} ?$

let $\bar{u} = \lim\limits_{r\to r_2} (r-r_2)\,u(r,t)$

$\Rightarrow L(\bar{u}) = 0$ Then $\bar{u} = C u_1$

it can not be u_2 because we know that u_2 will have log. singularity

Lemma 3

2. $\dfrac{\partial}{\partial r}(r-r_2)\,u(r,t)\Big|_{r=r_2} = \bar{u} \quad \Rightarrow L(\bar{u}) = 0$

$$u(r,z) = (z-z_0)^r \sum c_n(r)(z-z_0)^n$$

$$\frac{\partial}{\partial r}(r-r_2)\,u(r,z) = \frac{\partial}{\partial r}(z-z_0)^r \sum (r-r_2)c_n(r)(z-z_0^n)$$

$$= \log(z-z_0)\,u(r,z)(r-r_2)$$

$$+ (z-z_0)^r \sum \frac{\partial}{\partial r}\left[(r-r_2)c_n(r)\right](z-z_0$$

$$\bar{u} = u_2 = C u_1(z)\log(z-z_0) + (z-z_0)^{r_2}\sum B_n(r)(z-z_0)^n$$

$$B_n r = \lim_{r\to r_2}\frac{\partial}{\partial r}(r-r_2)c_n(r)$$

Lemma 4

3) $r_1 - r_2 = 0$

$$L(u) = (r-r_1)^2(z-z_0)^{r-2}c_0$$

$$\frac{\partial}{\partial r}u(r,z)\Big|_{r=r_2} = \bar{u} \qquad L(\bar{u})=0$$

$$= u_2$$

$$u_2(z) = u_1(z)\ln(z-z_0) + (z-z_0)^{r_1}\sum B_n(z-z_0)$$

$$B_n = \frac{\partial}{\partial r}c_n(r)\Big|_{r=r_1}$$

Lemma 2

a) $r_1 - r_2 = N$

$$u_2(z) = C\, u_1(z) \ln(z-z_0) + (z-z_0)^{r_2} \sum_{n=0} B_n (z-z_0)^n$$

$$B_n = \frac{d}{dr}\left[(r-r_2)\, c_n(r)\right]\Big|_{r=r_2}$$

b) $r_1 = r_2$

$$u_2(z) = c\, u_1(z) \ln(z-z_0) + (z-z_0)^{r_1} \sum_{n=1} B_n (z-z_0)^n$$

$$B_n = \frac{d}{dr} c_n(r)\Big|_{r=r_1}$$

Proof: $L(u) = \lambda_0(r)\,(z-z_0)^{r-2} c_0$
$= (r-r_1)(r-r_2)(z-z_0)^{r-2} c_0$

let $\bar{u} = \frac{d}{dr} (r-r_2) u \Big|_{r=r_2}$

$\Rightarrow L(\bar{u}) = 0$

$$u = (z-z_0)^r \sum c_n(r)\,(z-z_0)^n$$

$$(r-r_2)\, u(r) = (z-z_0)^r \sum (r-r_2)\, c_n(r)\,(z-z_0)^n$$

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$$\frac{d}{dr}(r-r_2)\,u(r) = \ln(z-z_0)\left((z-z_0)^r \sum (r-r_2)c_n (z-z_0)^n\right)$$

$$+ (z-z_0)^r \sum \left(\frac{d}{dr}(r-r_2)c_n\right)(z-z_0)^n$$

$$B_n = \frac{d}{dr}(r-r_2)\,c_n(r)$$

$$\bar{u} = u_2 = c\,u_1 \ln(z-z_0) + (z-z_0)^{r_2} \sum B_n (z-z_0)^n$$

$$\Rightarrow \lim_{r\to r_2} (r-r_2)c_n(r) = c_n(r_1)$$

b) if $r_1 = r_2$ $\Rightarrow L(u) = (r-r_1)^2 (z-z_0)^{r-2} c_0$

$$\Rightarrow \bar{u} = \frac{d}{dr}\, u(r)\Big|_{r=r_1}$$

$$L(\bar{u}) = 0$$

$$\bar{u} = \frac{d}{dr}(z-z_0)^r \sum c_n(r)(z-z_0)^n$$

$$= u_1 \ln(z-z_0) + (z-z_0)^{r_1} \sum B_n (z-z_0)^n$$

$$B_n = \frac{d}{dr} c_n(r)\Big|_{r=r_1}$$

Example $\quad \cancel{z}\, u'' - u = 0 \qquad a_0 = b_0 = 0$

$z_0 = 0 \qquad r_1 = 1, \; r_2 = 0$

$$u(z,r) = \sum c_n(r)\, z^{n+r}$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)\, c_n(r)\, z^{n+r-1} - \sum c_n(r)\, z^{n+r} = 0$$

$$r(r-1)\, c_0(0)\, z^{r-1} +$$

$$\sum_{n=0} \left[(n+r+1)(n+r)\, c_{n+1} - c_n\right] z^n = 0$$

$$c_{n+1}(r) = \frac{c_n(r)}{(n+r+1)(n+r)} \qquad n = 0, 1 \ldots$$

$$c_1 = \frac{c_0}{(r+1)\,r}$$

$$c_2 = \frac{1}{(r+2)(r+1)} \; \frac{c_0}{(r+1)\,r}$$

$$c_3 = \frac{1}{(r+3)(r+2)} \; \frac{1}{(r+2)(r+1)} \; \frac{c_0}{(r+1)\,r}$$

when $r = 1$.

$$c_1 = \frac{c_0}{1 \cdot 2}$$

$$c_2 = \frac{c_0}{3 \cdot 2^2 \, 1}$$

$$c_3 = \frac{c_0}{4 \cdot 3^2 \cdot 2^2 \cdot 1}$$

$$u_1(t) = z \sum_{k=0}^{\infty} \frac{z^k}{k!\,(k+1)!}$$

$$\boxed{c_k = c_0 \frac{1}{(k+1)!\,k!}}$$

$r_2 = 0$

$n(n+1) c_{n+1} = c_n$

$n=0 \qquad 0 = c_0 \neq 0$ Not const.

$\exists\, u_1(t)$ in general

$$c_n(r) = \frac{c_0}{(r+n)\left[(r+n-1)\cdots(r+1)\right]^2 n} \qquad n \geq 2$$

for the solut $r = r_2 = 0$.

$$\lim_{r\to 0} \frac{d}{dr} r\, c_n(r) = \lim_{r \to 0} \frac{d}{dr} \frac{c_0}{(r+n)[\ \ ]^2}$$

$$\log \frac{1}{(r+n)[\ ]^2} = -\log(r+n) - 2\log(r+n-1) - \cdots - 2\log(r+1)]$$

$$-\frac{d}{dr} \log \frac{1}{(r+n)[\ ]^2} = \left(\frac{1}{n} + \frac{2}{n-1} + \cdots + \frac{2}{1}\right) \frac{1}{(r+n)[\ ]^2}$$

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$$\frac{d}{dr} \log r c_n \Big|_{r=0} = \frac{\ln (r c_n)'}{r c_n} \Big|_{r=0} = \frac{1}{n} + \frac{2}{n-1} + \cdots + \frac{2}{1}$$

$$(r c_n)' \Big|_{r=0} = -\left(\frac{1}{n} + \frac{2}{n-1} + \cdots \frac{2}{1}\right)(r c_n)\Big|_{r=0}$$

$$= -\left(\frac{1}{n} + \frac{2}{n-1} + \cdots + \frac{2}{1}\right) c_n(r_1)$$

$$=$$

$$\varphi(n) = \begin{cases} \sum_{m=1}^{n} \frac{1}{m} \\ 0 & n=0 \end{cases}$$

$$\varphi(n-1) + \varphi(n) = \frac{1}{n} + \frac{2}{n-1} + \cdots + \frac{2}{1}$$

$$\Rightarrow \quad u_2(z) = C u_1(t) \log z + z^0 \sum_{n=0} \frac{\varphi(n-1) + \varphi(n)}{(n-1)!\, n!} z^n$$

$$= C u_1(t) \log z - 1 - \sum_{n=1}^{\infty} \frac{\varphi(n-1) + \varphi(n)}{n!\,(n-1)!} z^n$$

$$\vee$$

$(n-1)n[$

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Example: $z^2 u'' + z u' + z^2 u = 0$ $\qquad r_1 = r_2 = 0$

$z u'' + u' + z u = 0$

$a_0 = 1 \quad , \quad b_0 = 0 \qquad r(r-1) + r = 0$

$r_1 = r_2 = 0$

$u(z) = \sum c_n(r) z^{n+r}$

~~$\sum n(n-1) a_n(r)$~~

$$\sum (n+r)(n+r-1) c_n(r) z^{n+r-1} + \sum (n+r) c_n(r) z^{n+r-1} + \sum_{n=0}^{\infty} c_n(r) z^{n+r+1} = 0$$

$$\sum \left[(n+r)(n+r-1) + (n+r) \right] c_n z^{n+r-1} + \sum c_n(r) z^{n+r+1} = 0$$

$$\sum_{n=0} (n+r)^2 c_n z^{n-1} + \sum_{n=} c_n(r) z^{n+1} = 0$$

$$r^2 \frac{1}{z} c_0 + (1+r)^2 c_1 + \sum \left[(n+r+2)^2 c_{n+2} + c_n(r) \right] z^{n+}$$

$c_1 = 0, \quad , \qquad c_{n+2} = - \dfrac{c_n}{(n+r+2)^2}$

(D617)

$$c_{2n+1} = 0$$

$$c_{2n+2} = -\frac{c_{2n}}{(2n+r+2)^2} \qquad n=0.$$

$$c_2 = -\frac{c_0}{(r+2)^2}$$

$$c_4 = -\frac{c_2}{(r+4)^2} = \frac{c_0}{(r+4)^2(r+2)^2}$$

$r_1 = 0$

$$c_{2n}^{(1)} = \frac{(-1)^{n+1}}{2^{2n}(n!)^2}$$

$$c_{2n} = (-1)^{n+1} \frac{c_0}{[(r+2n)(r+2n-2)\cdots(r+2)]^2}$$

$$\ln c_{2n} = -\ln(r+2n) - \cdots$$

$$\frac{d}{dr} c_{2n} = -c_{2n}\left[\frac{1}{r+2n} + \cdots + \frac{1}{r+2}\right]$$

$$B_n = -c_{2n}(0) \cdot \frac{1}{2} H(n)$$

$$= +(-1)^n \frac{c_0}{2n\,(n!)^2} H(n)$$

$$u(z) = u_1(z)\ln z + \sum (-1)^n H(n) \frac{(z/2)^n}{(n!)^2}$$

$$u_1(z) = \sum \frac{(-1)^n (z/2)^n}{(n!)^2} = J_0(z)$$ Bessel's equation of order 0

SET 6

MATH 543: FROBENIUS METHOD

References: Hildebrand and Sadri Hassan.

The following theorem summarizes the Frobenius method: (Proved in Class please see your lecture notes and also DK)

Theorem 1. *Suppose that the differential equation $Lu = 0$, where L is a second order differential operator, has a regular singular point at $z = z_0$ with the roots r_1 and r_2 of the indical equation. There are three possible cases: (Assuming $Re(r_1) > Re(r_2)$)*

1. $r_1 - r_2 \neq$ an integer
2. $r_1 - r_2 = N$ (a non-negative integer) and recursion relation is consistent
3. $r_1 - r_2 = N$ (a non-negative integer) and recursion relation is not consistent.

Then, in the first two cases, there exists a bases of $\{u_1, u_2\}$ of solutions of $Lu = 0$. These solutions are of the form

$$u_1(z) = (z - z_0)^{r_1} \sum_{k=0}^{\infty} C_k^1 (z - z_0)^k, \tag{1}$$

$$u_2(z) = (z - z_0)^{r_2} \sum_{k=0}^{\infty} C_k^2 (z - z_0)^k \tag{2}$$

and the third case , the bases $\{u_1, u_2\}$ is of the following form

$$u_1(z) = (z - z_0)^{r_1} \sum_{k=0}^{\infty} C_k (z - z_0)^k, \tag{3}$$

$$u_2(z) = C u_1(z) \ln(z - z_0) + (z - z_0)^{r_2} \sum_{k=1}^{\infty} B_k (z - z_0)^k \tag{4}$$

where the power series about $(z - z_0)$ are convergent in a neighborhood of $z = z_0$

A method when $r_1 - r_2 = N$

Let $z = z_0$ be a regular singular point of differential equation

$$Lu = u'' + p(z)\, u' + q(z)u = 0, \tag{5}$$

where

$$A(z) = (z - z_0)\, p(z), \quad B(z) = (z - z_0)^2\, q(z) \tag{6}$$

are analytic in a neighborhood D of $z = z_0$ hence $A(z)$ and $B(z)$ are analytic in D then

$$A(z) = \sum_{k=0}^{\infty} a_k\, (z - z_0)^k, \quad B(z) = \sum_{k=0}^{\infty} b_k\, (z - z_0)^k, \tag{7}$$

Using the anzats

$$u(z) = (z - z_0)^r \sum_{k=0}^{\infty} C_k\, (z - z_0)^k, \tag{8}$$

in the DE we get

$$\lambda_0(n + r)\, C_n + \sum_{m=0}^{n-1} [(m + r)a_{n-m} + b_{n-m}]\, C_m, \tag{9}$$

This is the recursion relations mentioned in the above theorem 1. Here

$$\lambda_0(r) = r(r - 1) + r\, a_0 + b_0 \tag{10}$$

If the indical equation is not imposed $\lambda_0(r) \neq 0$ and the remaining

$$C_1(r), C_2(r), \cdots, C_n(r) \tag{11}$$

are determined through the recursion relation. Since $n = 0$ term is not set to zero then the differential equation is not satisfied but becomes

$$Lu = C_0\, \lambda_0(r)\, (z - z_0)^{r-2} \tag{12}$$

where C_0 is the undetermined constant. The solution of this equation is

$$u(r, z) = (z - z_0)^r \sum_{k=0}^{\infty} C_k(r)\, (z - z_0)^k, \tag{13}$$

In the case 3 , when the difference of the indices is a nonegative number and recursion relations are inconsistent the the usual anzats fails. To find the correct anzats and a method of solution we use (12) and (13).

$r_1 = r_2$ **case**
Eq.(12) becomes

$$Lu = C_0\,(r - r_1)^2\,(z - z_0)^{r-2}, \tag{14}$$

Taking the derivative of both sides wrt r and letting $r = r_1$ we obtain the second solution as

$$u_2(z) = \frac{du(r,z)}{dr}|_{r=r_1}, \tag{15}$$

Using the form of $u(r,z)$ given in (13) we obtain

$$u_2(z) = u_1(z)\,\ln(z - z_0) + (z - z_0)^{r_1} \sum_{k=0}^{\infty} C_k'(r_1)\,(z - z_0)^k \tag{16}$$

where $C_k'(r_1) = \frac{d}{dr}\,C_k(r)\,|_{r=r_1}$

$r_1 - r_2 = N$ **a positive integer case**
Eq.(12) becomes

$$Lu = C_0(r - r_1)(r - r_2)\,(z - z_0)^{r-2} \tag{17}$$

Multiplying both sides by $r - r_2$ and taking derivatives of both sides wrt r and taking limit as r goes to r_2 we obtain

$$u_2(z) = \frac{d}{dr}\,[(r - r_2)\,u(r,z)] \;\text{ at } r = r_2 \tag{18}$$

Using the expression (13) for $u(r,z)$ we obtain

$$u_2(z) = Cu_1(z)\,\ln(z - z_0) + (z - z_0)^{r_2} \sum_{k=0}^{\infty} \tilde{C}_k\,(z - z_0)^k \tag{19}$$

where

$$\tilde{C}_k = \frac{d}{dr}[(r - r_2)\, C_k(r)]\,|_{r=r_2}$$

Problems

1. An example for the case $r_1 - r_2 = N$ with consistent recursion relations. Solve $z^2u'' + (z^2 + z)u' - u = 0$ about $z = 0$ (solved in Class)

2. An example for the case $r_1 = r_2$ with inconsistent recursion relations. Solve $z^2u'' + zu' + z^2u = 0$ about $z = 0$. You can find the solution u_1 by using the recursion relations. This solution is known as the Bessel function of order 0,

$$u_1(z) = J_0(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{(k!)^2}$$

The second solution $u_2(z)$ is found by using (16). Prove that it takes the form

$$u_2(z) = J_0(z)\, \ln z + \sum_{k=0}^{\infty} \varphi(k) \frac{(z/2)^{2k}}{(k!)^2}$$

where

$$\varphi(k) = \begin{cases} \sum_{m=1}^{k} \frac{1}{m}, & \text{for } k = 1, 2, \cdots \\ 0, & \text{for } (k = 0) \end{cases} \tag{20}$$

3. An example for the case $r_1 - r_2 = N$ with inconsistent recursion relations. Solve $z\, u'' - u = 0$ about $z = 0$. Again $u_1(z)$ will be found by the standard method

$$u_1(z) = \sum_{k=0}^{\infty} \frac{z^{k+1}}{k!(k+1)!}$$

And the second solution is found by using the formula (19)

$$u_2(z) = u_1\, \ln z + 1 - \sum_{k=1}^{\infty} \frac{\varphi(k) + \varphi(k-1)}{k!(k-1)!} z^k$$

4. Find the solutions of the Bessel's equation $z^2u'' + zu' + (z^2 - p^2)u = 0$. Solutions of this equation are known as the Bessel's functions of order p. Solve this equation for $p =$ not integer and for $p =$ integer.

5. Show that for the differential equation $z\,u'' + 3u' + 4zu = 0$ the condition $u(0) = 1$ determines a unique solution, and that $u'(0)$ cannot also be prescribed. Determine this solution.

6. Use the method of Frobenius to obtain the general solution of each of the following differential equations, valid near $z = 0$:

(1) $z^2u'' - 2zu' + (2 - z^2)u = 0$
(2) $(z - 1)u'' - zu' + u = 0$
(3) $2zu'' + (1 - 2z)u' - u = 0$
(4) $z^2u'' + zu' + (z^2 - \frac{1}{4})u = 0$
(5) $zu'' - u' + 4z^3u = 0$
(6) $zu'' + 2u' + zu = 0$
(7) $z(1 - z)u'' - 2u' - 2 + 2u = 0$

7. Determine the two values of the constant α for which it is true that all solutions of the equation $zu'' + (z - 1)u' - \alpha u = 0$ are regular at $z = 0$, and obtain the general solution in each of these cases

8. (a) Show that the equation $zu'' + u' - u = 0$ possesses equal indices $r_1 = r_2 = 0$ at $z = 0$
(b) Obtain the regular solution $u_1(z)$
(c) Assume the second solution of the form $u_2(z) = Cu_1(z)\ln z + v(z)$ where $C \neq 0$. Find $v(z)$.

9. (a) Show that the equation $zu'' - zu' - u = 0$ possesses indices $r_1 = 1, r_2 = 0$ at $z = 0$.
(b) Obtain the regular solution $u_1(z)$
(c) Assume the general form for the second solution as $u_2(z) = Cu_1 \ln z + v(z)$, where $C \neq 0$. Find $v(z)$.

Lecture 10

- Fuchsian Differential Equations

Fuchsian DE

(i) Fuchsian DE with two regular singular point.

(ii) Fuchsian DE with three regular singular points. (Hypergeometric Function)

(iii) Fuchsian DE with two regular singular point one of order two. (Confluent hypergeometric functions).

$$u'' + \frac{a_0}{z-a_1} u'' + \frac{b_0}{(z-z_1)^2} u = 0$$

Euler type.

$z_1 \neq 0$.

Fuchsian Differential Equations

Definition 1: A homogeneous DE with single-valued analytic coefficient is called "Fuchsian" differential DE if it has only regular singular points in the extended complex plane, i.e., the complex plane including the point at infinity.

Definition 2 (behavior at infinity). Let $z = 1/t$

$u'' + p(z) u' + q(z) u = 0$ changes to, with $v(t) = u(1/t)$

$v'' + \left[\frac{2}{t} - \frac{1}{t^2} r(t)\right] v' + \frac{1}{t^4} s(t) v = 0$ where $r(t) = p(1/t)$

$s(t) = q(1/t)$. Reao

$$u_z = - v_t \frac{1}{z^2}, \quad u_{zz} = \frac{2 v_t}{z^3} + v_{tt}\frac{1}{z^4} = 2t^3 v_t + t^4 v_{tt}$$

$$\cancel{2t^3 v_t} + t^4 v_{tt} - p(1/t) v_t t^2 + q(1/t) v = 0$$

$$v_{tt} + \left[\frac{2}{t} - \frac{1}{t^2} r(t)\right] v_t + \frac{1}{t^4} s(t) v = 0$$

We Assume that $t = 0$ is a regular singular point

hence

$r(t) = a_1 t + a_2 t^2 + \ldots$

$s(t) = b_2 t^2 + b_3 t^3 + \ldots$

$p(z) = \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots$

$q(z) = \frac{b_2}{z^2} + \frac{b_3}{z^3} + \ldots$

$$P(\xi) = -2\frac{z-z_2}{z_1-z_2} + \frac{(z-z_2)^2\, p(z)}{z_1-z_2}$$

$$z\xi - \xi z_2 = z - z_1$$

$$z(\xi-1) = \xi z_2 - z_1 \qquad z = \frac{\xi z_2 - z_1}{\xi - 1}$$

$z = z_2 \;:\; \xi = \infty$

$z = z_1 \;:\; \xi = 0$

$$z - z_2 = \frac{\xi z_2 - z_1}{\xi - 1} - z_2 = \frac{\xi z_2 - z_1 - z_2\xi + z_2}{\xi - 1} = \frac{z_2 - z_1}{\xi - 1}$$

$$P(\xi) = \frac{2}{\xi-1} + (z_1 - z_2)\frac{1}{(\xi-1)^2}\, p$$

$$Q(\xi) = \frac{(z_1-z_2)^2}{(\xi-1)^4}\, q(z)$$

$$u_{\xi\xi} + P(\xi)\, u_\xi + Q(\xi)\, u = 0$$

$\xi = \infty$ is the regular singular point

$$P(\xi) = \frac{a_1}{\xi} + \frac{a_2}{\xi^2} + \dots$$

$$Q(\xi) = \frac{b_2}{\xi^2} + \frac{b_3}{\xi^3} + \dots$$

$\xi = 0$ is also regular singular $\qquad \xi P(\xi) = \text{analy}$

$\qquad \xi^2 Q(\xi) = \text{analy}$

$a_2 = 0,\; a_3 = 0 \;\dots$

$b_3 = 0,\; b_4 = 0 \;\dots$

$$u_{\xi\xi} + \frac{a_1}{\xi} u_\xi + \frac{b_2}{\xi^2} u = 0$$

Euler type.

Proposition 1. A second order Fuchsian DE with two regular singular points can be reduced to a DE with constant coefficients.

Proof: let $z = z_1$ and $z = z_2$ be the regular singular points of the DE. Define now

$$\xi = \frac{z - z_1}{z - z_2}$$

then the DE (1) reduces to

$$u_{\xi\xi} + P(\xi)\, u_\xi + Q(\xi)\, u = 0$$

$$u_z = u_\xi \left(\frac{z - z_2 - z + z_1}{(z - z_2)^2} \right) = u_\xi \, \frac{(z_1 - z_2)}{(z - z_2)^2}$$

$$u_{zz} = u_{\xi\xi} \frac{(z_1 - z_2)^2}{(z - z_2)^4} - 2 u_\xi \frac{z_1 - z_2}{(z - z_2)^3}$$

$$u_{\xi\xi} \frac{(z_1 - z_2)^2}{(z - z_2)^4} - 2 u_\xi \frac{z_1 - z_2}{(z - z_2)^3} + p(z) \frac{z_1 - z_2}{(z - z_2)^2} u_\xi + q(z)\, u = 0$$

$$u_{\xi\xi} + P(\xi)\, u_\xi + Q(\xi)\, u = 0$$

$$\frac{(z - z_2)^2}{(z_1 - z_2)^2} \left[-2 \frac{z_1 - z_2}{z - z_2} + p\,(z_1 - z_2) \right] = P(\xi)$$

$$\frac{(z - z_2)^4}{(z_1 - z_2)^2}\, q(z) = Q(\xi)$$

Regular singular points with three regular singul. points

$$u'' + \left[\frac{1-\alpha-\alpha'}{z-z_1} + \frac{1-\beta-\beta'}{z-z_2} + \frac{1-\gamma-\gamma'}{z-z_3} \right] u'$$

$$+ \left[\frac{(z_1-z_2)(z_1-z_3)}{z-z_1}\alpha\alpha' + \frac{(z_2-z_1)(z_2-z_3)}{z-z_2}\beta\beta' + \frac{(z_3-z_1)(z_3-z_2)}{z-z_3}\gamma\gamma' \right] \frac{u(z)}{(z-z_1)(z-z_2)(z-z_3)} = 0$$

$$\alpha+\alpha'+\beta+\beta'+\gamma+\gamma' = 1$$ "Riemann equation"

$z=z_1 \quad : \quad r_1=\alpha, \; r_2=\alpha'$

$z=z_2 \quad : \quad r_1=\beta, \; r_2=\beta'$

$z=z_3 \quad : \quad r_1=\gamma, \; r_2=\gamma'$

$$u(z) = P\left\{ \begin{matrix} z_1 & z_2 & z_3 & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{matrix} \right\}$$ "Riemann P symbol"

The 9 parameters which characterize the Riemann equation. These parameters can be reduced to 3 by making use of the following transformations on the equation

(i) $u(z) = (z-z_1)^r (z-z_2)^s (z-z_3)^t \, v(z)$

with $r+s+t=0$

(ii) $$z' = \frac{Az+B}{Cz+D}$$

<u>Proof.</u>

$$
\begin{aligned}
u' = {} & r(z-z_1)^{r-1}(z-z_2)^s(z-z_3)^t v \\
& + s(z-z_1)^r(z-z_2)^{s-1}(z-z_3)^t v \\
& + t(z-z_1)^r(z-z_2)^s(z-z_3)^{t-1} v \\
& + (z-z_1)^r(z-z_2)^s(z-z_3)^t v'
\end{aligned}
$$

$$
\begin{aligned}
u'' = {} & r(r-1)(z-z_1)^{r-2}(z-z_2)^s(z-z_3)^t v \\
& + s(s-1)(z-z_1)^r(z-z_2)^{s-2}(z-z_3)^t v \\
& + t(t-1)(z-z_1)^r(z-z_2)^s(z-z_3)^{t-2} v \\
& + 2rs(z-z_1)^{r-1}(z-z_2)^{s-1}(z-z_3)^t v \\
& + 2rt(z-z_1)^{r-1}(z-z_2)^s(z-z_3)^{t-1} v \\
& + 2st(z-z_1)^r(z-z_2)^{s-1}(z-z_3)^{t-1} v \\
& + 2r(z-z_1)^{r-1}(z-z_2)^s(z-z_3)^t v' \\
& + 2s(z-z_1)(z-z_2)^{s-1}(z-z_3)^t v' \\
& + 2t(z-z_1)(z-z_2)^s(z-z_3)^{t-1} v' \\
& + (z-z_1)^r(z-z_2)^s(z-z_3)^t v''
\end{aligned}
$$

$$
u'' + \left[\frac{1-\alpha-\alpha'}{z-z_1} + \frac{1-\beta-\beta'}{z-z_2} + \frac{1-\gamma-\gamma'}{z-z_3}\right]u'
$$

⑥

$$v'' + \left[\frac{2\cancel{r}}{z-z_1} + \frac{2s}{z-z_2} + \frac{2t}{z-z_3}\right] v'$$

$$+ \left[\frac{1-\alpha-\alpha'}{z-z_1} + \frac{1-\beta-\beta'}{z-z_2} + \frac{1-\gamma-\gamma'}{z-z_3}\right] v'$$

$$+ \frac{r(r-1)}{(z-z_1)^2} + \frac{s(s-1)}{(z-z_2)^2} + \frac{t(t-1)}{(z-z_3)^2} + \frac{2rs}{(z_1-z_1)(z-z_2)}$$

$$+ \frac{2rt}{(z-z_1)(z-z_3)} + \frac{2st}{(z-z_2)(z-z_3)}$$

$$+ \left[\frac{1-\alpha-\alpha'}{z-z_1} + \frac{1-\beta-\beta'}{z-z_2} + \frac{1-\gamma-\gamma'}{z-z_3}\right]\left(\frac{r}{z-z_1} + \frac{s}{z-z_2} + \frac{t}{z-z_3}\right) v$$

$$+ \left[\qquad\qquad\qquad\right] \frac{v}{(z-z_1)(z-z_2)(z-z_3)} = 0$$

$$v'' + \left[\frac{1-(\alpha\mp r)-(\alpha'\mp r)}{z-z_1} + \frac{1-(\beta\mp r)-(\beta'\mp r)}{z-z_2} + \frac{1-(\gamma\mp r)-(\gamma'\mp r)}{z-z_3}\right] v$$

$$v(z) = P\begin{Bmatrix} z_1 & z_2 & z_3 \\ \alpha\mp r & \beta\mp r & \gamma\mp r \\ \alpha'\mp r & \beta'\mp r & \gamma'\mp r \end{Bmatrix}$$

$$\Rightarrow \quad P\begin{Bmatrix} z_1 & z_2 & z_3 \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' \end{Bmatrix} = (z-z_1)^r (z-z_2)^s (z-z_3)^t \, P\begin{Bmatrix} z_1 & z_2 & z_1' \\ \alpha\mp r & \beta\mp s & \gamma\mp t \\ \alpha'\mp r & \beta'\mp s & \gamma'\mp t \end{Bmatrix}$$

$$= (z-z_1)^r (z-z_2)^s (z-z_3)^t \, P\begin{Bmatrix} z_1 & z_2 & z_3 \\ \alpha-r & \beta-s & \gamma-t \\ \alpha'-r & \beta'-s & \gamma'-t \end{Bmatrix}$$

or

$$P\left\{\begin{matrix} z_1 & z_2 & z_3 & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{matrix}\right\} = (z-z_1)^r (z-z_2)^s (z-z_3)^t \, P\left\{\begin{matrix} z_1 & z_2 & z_3 & \\ \alpha-r & \beta-s & \gamma-t & z \\ \alpha'-r & \beta'-s & \gamma'-t & \end{matrix}\right\}$$

($r \to -r$, $s \to -s$, $t \to -t$)

or

$$P\left\{\begin{matrix} z_1 & z_2 & z_3 & \\ \alpha+r & \beta+s & \gamma+t & z \\ \alpha'+r & \beta'+s & \gamma'+t & \end{matrix}\right\} = (z-z_1)^r (z-z_2)^s (z-z_3)^t \, P\left\{\begin{matrix} z_1 & z_2 & z_3 & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{matrix}\right\}$$

$r+s+t=0$

(ii) by using the Mobius transformation $z' = \dfrac{Az+B}{Cz+D}$ with $AD-BC \neq 0$ then

$$P\left\{\begin{matrix} z_1' & z_2' & z_3' & \\ \alpha & \beta & \gamma & z' \\ \alpha' & \beta' & \gamma' & \end{matrix}\right\} = P\left\{\begin{matrix} z_1 & z_2 & z_3 & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{matrix}\right\}$$

where $z_i' = \dfrac{Az_i+B}{Cz_i+D}$, $i=1,2,3$.

let z_1', z_2' and z_3' be respectively $0, \infty$ and 1.

hence $z_1' = 0$ $\quad 0 = Az_1 + B$, $\dfrac{z_2' = \infty}{Cz_2 + D = 0}$, $1 = \dfrac{Az_3 + B}{Cz_3 + D}$

$$\frac{C}{D} = -\frac{1}{z_2} , \quad B = -Az_1 ,$$

$$z_3' = 1 \quad 1 = \frac{Az_3 - Az_1}{Cz_3 + D} = \frac{A}{D}\left(\frac{z_3 - z_1}{1 - z_3/z_2}\right)$$

$$\rightarrow \frac{A}{D} = \frac{z_2 - z_3}{z_2(z_3 - z_1)} , \rightarrow \frac{C}{D} = -\frac{1}{z_2}$$

$$\rightarrow \frac{B}{D} = -\frac{A}{D} z_1 = -\frac{z_1(z_2 - z_3)}{z_2(z_3 - z_1)}$$

$$\Downarrow$$

$$\Rightarrow z' = \frac{Az + B}{Cz + D} = \frac{\frac{A}{D} z + \frac{B}{D}}{\frac{C}{D} z + 1} = \frac{(z_2 - z_3) z - z_1(z_2 - z_3)}{-(z_3 - z_1) z + z_2(z_3 - z_1)}$$

$$z' = \frac{(z_2 - z_3)(z - z_1)}{(z_3 - z_1)(-z + z_2)}$$

Choosing r, s, t properly we simplify the solution

$$P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ \alpha & \beta & \gamma, & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} = (z-z_1)^{-r}(z-z_2)^{-s}(z-z_3)^{-t}P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ \alpha+r & \beta+s & \gamma+t & z \\ \alpha'+r & \beta'+s & \gamma'+t & \end{Bmatrix}$$

$$r = -\alpha, \quad s = \alpha+\gamma, \quad t = -\gamma$$

$$P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ \alpha, & \beta, & \gamma, & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} = (z-z_1)^{\alpha}(z-z_2)^{-\alpha-\gamma}(z-z_3)^{\gamma}P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ 0 & \beta+\alpha+\gamma & 0 & z \\ \alpha'-\alpha & \beta'+\alpha+\gamma & \gamma'-\gamma & \end{Bmatrix}$$

$$= \left(\frac{z-z_1}{z-z_2}\right)^{\alpha}\left(\frac{z-z_3}{z-z_2}\right)^{\gamma}P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ 0 & \overset{(a)}{\alpha+\beta+\gamma} & 0 & z \\ \underset{(1-c)}{\alpha'-\alpha} & \overset{(b)}{\alpha+\beta'+\gamma} & \underset{(c-a-b)}{\gamma'-\gamma} & \end{Bmatrix}$$

$\because a = \alpha+\beta+\gamma$
$b = \alpha+\beta'+\gamma$
$1-c = \alpha'-\alpha$
$c-a-b = \gamma'-\gamma$
$1 = 1$

$z \to z'$

$$= \left(\frac{z-z_1}{z-z_2}\right)^{\alpha}\left(\frac{z-z_3}{z-z_2}\right)^{\gamma}P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & \alpha+\beta+\gamma & 0 & z' \\ \alpha'-\alpha & \alpha+\beta'+\gamma & \gamma'-\gamma & \end{Bmatrix}$$

where

$$z' = \frac{(z_3-z_2)(z-z_1)}{(z_3-z_1)(z-z_2)}$$

let $\quad a = \alpha+\beta+\gamma, \quad b = \alpha+\beta'+\gamma, \quad c = 1+\alpha-\alpha'$

$$\Rightarrow P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} = \left(\frac{z-z_1}{z-z_2}\right)^{\alpha}\left(\frac{z-z_3}{z-z_2}\right)^{\gamma}P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & a & 0 & z' \\ 1-c & b & c-a-b & \end{Bmatrix}$$

0) Rieman Eqn $\rightarrow$ hypergeometric eqn

1) Solutions about the singular points

2) Kummer's Solution

3) Find the solutions about $z=1$ and $z=\infty$ by using the Kummer's solutions.

$$\Phi \begin{Bmatrix} z_1 & z_2 & z_3 \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{Bmatrix} = (\ldots$$

4) Special cases.

5) Confluent Hypergeometric function: $zu'' + (c-z)u' - au$

a) $\Phi(a,b;z) = \lim_{b\to\infty} F(a;b;c;z/b)$

b) $z=1 \rightarrow z=\infty$ $\quad z=\infty$ not regular

c) Series soln.

d) Functions related to CHGF

$a = \alpha + \beta + \gamma$

$b = \alpha + \beta' + \gamma$

$1 - c = \alpha' - \alpha$

$$P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & a & 0 & z' \\ 1-c & b & c-a-b & \end{Bmatrix}$$

$\Rightarrow$

$$u'' + \left[\frac{1-1+c}{z} + \frac{1-c+a+b}{z-1} \right] u' + ab\,u = 0$$

$$z(z-1)u'' + \left[c(z-1) + z(1-c+a+b) \right] u' + ab\,u = 0$$

$$z(z-1)u'' + \left[-c + z(c+1-c+a+b) \right] u' + ab\,u = 0$$

$$\underline{z(1-z)u'' + \left[c - z(1+a+b) \right] u' - ab\,u = 0}$$

$a = \alpha + \beta + \gamma$

$b = \alpha + \beta' + \gamma$

$1 - c = \alpha' - \alpha$

DE for the reduced equation is

$$-\text{var} \quad z(z-1)u'' + [c-(a+b+1)z]u' - abu = 0$$

$$z(1-z)$$

$z_2 = \infty$, $z_1 = 0$, $z_3 = 1$ $\quad$ ($\alpha = 0$, $\gamma = 0$, $\alpha' = 1-c$, $\gamma' = c-a-b$, $\beta = a$, $\beta' = b$

$$u'' + \left(\frac{1-\alpha-\alpha'}{z} + \frac{1-\gamma-\gamma'}{z-1}\right)u' + \frac{1}{z(z-1)}\,ab\cdot u$$

$$\cancel{z(z-1)u'' + [\alpha+\alpha'-1+z(1-\gamma-\gamma'+1-\alpha-\alpha')]u + abu = 0}$$

$$u'' + \left(\frac{1-1+c}{z} + \frac{1-c+a+b}{z-1}\right)u' + \frac{ab}{z(z-1)}u = 0$$

$$z(z-1)u'' + [-c + z(1+a+b)]u' + abu = 0$$

$$\Rightarrow \quad z(1-z)u'' + [c - z(1+a+b)]u' - abu = 0$$

Hypergeometric equ.

$$\Rightarrow \quad u(z) = F(a,b;c;z)$$

$$P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} = \left(\frac{z-z_1}{z-z_2}\right)^{\alpha}\left(\frac{z-z_3}{z-z_2}\right)^{\gamma} F(a;b;c,z')$$

$$F\left(\alpha+\beta+\gamma,\ \alpha+\beta'+\gamma,\ 1+\alpha-\alpha';\ \frac{(z-z_1)(z_3-z_2)}{(z-z_2)(z_3-z_1)}\right)$$

Hypergeometric Function

$$z(1-z)u'' + [c-(a+b+1)z]u' - ab\,u = 0$$

regular singular pts

$z=0$, $r_1=0$, $r_2=1-c$

$z=\infty$ $r_1=a$, $r_2=b$.

$z=1$ $r_1=0$, $r_2=\gamma'-\gamma = c-a-b$.

~~Regular~~ solutions about the regular singular pts.

1) $z=0$, $r_1=0$

$$F(a,b;c;z) = \sum_{n=0}^{\infty} c_n z^n$$

$$c_n = \frac{(a+n-1)(b+n-1)}{n(n+c-1)} c_{n-1} \qquad n=1,2,\dots$$

when c is ~~non-positive~~ not equal to $0,-1,-2,\dots$

we have

$$c_n = \frac{(a+n-1)(b+n-1)}{n(n+c-1)} c_{n-1} = \frac{a(a+1)\cdots(a+n-1)\,b(b+1)\cdots(b+n-1)}{n!\,c(c+1)\cdots(c+n+1)}$$

$$= \frac{\Gamma(a+n)/\Gamma(a) \quad \Gamma(b+n)/\Gamma(b)}{\Gamma(c+n)/\Gamma(c) \quad \Gamma(n+1)} \qquad c_0 = 1$$

hence

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}$$

$$\left[+ c(1-c) - ab + \frac{-(1-c)\,c - (a+b+1)}{z} \right] g_1$$

$$\cancel{c^2 - (a+b+1)}$$

$$\Rightarrow \quad z(1-z)\,g_1'' - c(1-c)\,\frac{1-z}{\cancel{z}}\,g_1 + 2(1-c)(1-z)\,g_1'$$

$$+ (1-c)\left[\frac{\cancel{c} - (a+b+1)z}{z}\right] g_1 + \left[c - (a+b+1)z\right] g_1'$$

$$- ab\, g_1 = 0$$

$$z(1-z)\,g_1'' + \left[2(1-c) + c - (a+b+1+2-2c)z\right] g_1'$$

$$+ \left[c(1-c) - (a+b+1) - ab\right] g_1 = 0$$

$$z(1-z)\,g_1'' + \left[2-c - (a+b+3-2c)z\right] g_1'$$

$$\bar{\text{(+)}} \left[a-c+1\right]\left[b-c+1\right] g_1 = 0$$

$$\checkmark$$

$$g_1 \sim \cancel{F(a,b,c;z)}$$

$$z(1-z)\,g_1'' + \left[2-c - (a-c+1+b-c+1+1)z\right] g_1'$$

$$\bar{\text{(+)}} (a-c+1)(b-c+1)\,g_1 = 0$$

$$F(a-c+1, b-c+1; 2-c; z) \quad \checkmark$$

$$\checkmark$$

$$\checkmark \quad u(z) = \alpha\, F(a,b;c;z) + \beta\, z^{1-c}\, F(a-c+1, b-c+1; 2-c; z)$$

This is called the "hypergeometric" series

ex $$F(1,b,b;z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

" Geometric series"

The above series solution is valid up to next singular point $z=1$. Hence

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!} \qquad |z|<1$$

second soln. $r = r_2 = 1-c$ $\Rightarrow$ (13) den sonraki sahifelere bak

If $1-c$ is NOT an integer, a second solution of (1) is of the form $z^{1-c} g_1(z)$ where $g_1(z)$ satisfies

$$z(z-1) g_1'' + [(a+b-2c+3)z + c-2] g_1' + (a-c+1)(b-c+1) g_1 = 0$$

!!

Proof. $u' = (1-c) z^{-c} g_1 + z^{1-c} g_1'$

$u'' = -c(1-c) z^{-c-1} g_1 + 2(1-c) z^{-c} g_1' + z^{1-c} g_1''$

$\cancel{z^{1-c}} g_1'' z(1-z) - c\,z(1-z) z^{-c-1}(1-c) g_1 + 2z(1-z)(1-c)\cancel{z^{-c}} g_1'$

$+ [c-(a+b+1)z](1-c) z^{-c} g_1 + [c-(a+b+1)z]\cancel{z^{1-c}} g_1'$

$- ab\, z^{1-c} g_1 = 0$

$z(1-z) g_1'' + [2(1-z)(1-c) + c - (a+b+1)z] g_1'$

$+ [-\frac{1-z}{z} c(1-c) + \frac{c-(a+b+1)}{z} - ab] g_1 = 0$

Second regular singular point

Similarly

for $z=1$ $\quad r_1 = 0, \quad r_2 = c-a-b.$

we have if $c-a-b =$ NOT integer

$$u(z) = g_2(1-z) + (1-z)^{c-a-b}\, g_3(1-z)$$

where g_2 and g_3 power series of $1-z$.

for $z=\infty$ $\quad r_1 = a, \quad r_2 = b$

we have $a-b$ = NOT integer

$$u(z) = \left(\frac{1}{z}\right)^a g_4\left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^b g_5\left(\frac{1}{z}\right)$$

where g_4 and g_5 are power series in $(1/z)$.

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\,\Gamma(b+n)\,z^n}{\Gamma(c+n)\,n!} \qquad (1).$$

is valid only up to the next singul. point $z=1$ hence the above series expansion is valid for $|z|<1$

ii) $r_2 = 1-c$. we assume that $1-c$ = NOT a non-negative integer.

then $u_2(z) = z^{1-c}\, g_1(z)$

insert it into the HDE

$$z(1-z)u'' + [c-(1+a+b)z]u' - abu = 0$$

we get

$$z(z-1)g_1'' + [c-2+(a+b-2c+3)z]g_1' + (a-c+1)(b-c+1)g_1 = 0$$

or

$$z(1-z)g_1'' + [2-c-(a+b-2c+3)z]g_1' - (a-c+1)(b-c+1)g_1$$

$$\Rightarrow g_1(z) = F(a-c+1,\, b-c+1;\, 2-c;\, z)$$

$1+a+b-2c+2$

$$\Rightarrow u(z) = A\,F(a,b;c;z) + B\,z^{1-c}\,F(a-c+1,\, b-c+1;\, 2-c;\, z)$$

2) about the regular singular pt $z = \cancel{0}\ 1$

$r_1 = 0$, $r_2 = c-a-b$ = NOT an integer.

~~u(z)~~ =

Hypergeom. eqn

$$z(1-z)u'' + [c-(1+a+b)z]u' - ab\,u = 0$$

let $z = 1-y \Rightarrow 1-z = 1-1+y = y$

$u = g_2(y)$

$$(1-y)y\,u_{yy} + [c-(1+a+b)(1-y)]u_y - ab\,u = 0$$

$$(1-y)y\,u_{yy} + [-c+1+a+b + (1+a+b)y]u_y - ab\,u = 0$$

$$u_1 = g_2(z) = F(a,b;-c+1+a+b;\ 1-z)$$

$$u_2 = (1-z)^{c-a-b}\, g_3(1-z)$$

$$g_3 = F(c-b,\ c-a;\ 1+c-a-b;\ 1-z)$$

$$u(z) = A\,F(a,b;-c+1+a+b;\ 1-z) + B(1-z)^{c-a-b}F(c-b,c-a;1+c-a-b;\ 1-z)$$

3) about the regular singular pt $z = \infty$

i) $r_1 = a$ $\quad$ $a - b$ = NOT an integer

$$u_1(z) = z^{-a} g_4(1/z)$$

let $u(z) = z^r g_4(1/z)$ then the HDG becomes

$$y(1-y) g_4'' + [\cancel{1} a - b - 2r - (2 - c - 2r) z] g_4'$$
$$+ [r^2 - ar + rc - \tfrac{1}{z}(r+a)(r+b)] g_4 = 0$$

$r = -a$

$$y(1-y) g_4'' + [1 + a - b - (2 + 2a - c) z] g_4'$$
$$+ a(-1 - a + c) g_4$$

$$g_4 = F(a, 1 + a - c; 1 + a - b; \tfrac{1}{z})$$

$$a^2 + a - ac$$
$$-a(-a + 1 + c$$

Similarly

$$u(z) = A z^{-a} F(a, 1 + a - c; 1 + a - b; \tfrac{1}{z})$$
$$+ B z^{-b} F(b, 1 + b - c; 1 + b - a; \tfrac{1}{z})$$

Kummer's solution for F

$$u(z) = P\left\{\begin{matrix} z_1 & z_2 & z_3 & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{matrix}\right\}$$

$$= \left(\frac{z-z_1}{z-z_2}\right)^{\alpha}\left(\frac{z-z_3}{z-z_2}\right)^{\gamma} F\Big(\underbrace{\alpha+\beta+\gamma}_{a},\ \underbrace{\alpha+\gamma+\beta'}_{b};\ \underbrace{1+\alpha-\alpha'}_{c};\ \frac{(z-z_1)(z_3-z_2)}{(z-z_2)(z_3-z_1)}\Big) \quad (15.1)$$

(i) collums of P can be interchanged

6 New solutions

1 2 3	3 1 2
1 3 2	3 2 1
2 1 3	
2 3 1	

6 differ

$a \leftrightarrow b$ invariance of F

(ii) Riemann eqn. is invariant for $\alpha \leftrightarrow \alpha'$, $\beta \leftrightarrow \beta'$, $\gamma \leftrightarrow \gamma'$

but (15.1) is invariant here

$\alpha \leftrightarrow \alpha'$, $\gamma \leftrightarrow \gamma'$

gives two transformation for each we have two solns

as a total 4 new solutions.

Hence $4 \times 6 = 24$ new solutions "Kummer's Solutions".

α	γ	α	γ'
α'	γ'	α'	γ
α'	γ	α'	γ'
α	γ'	α	γ

4 differ

Alternative way to find solution about z=1 using the symmetries

Multiply the right handside by $(-1)^{\alpha} z_2^{\alpha+\gamma}$

and interchange the first and last columns.

$z_3 \leftrightarrow z_1$, $\alpha \leftrightarrow \gamma$, $\alpha' \leftrightarrow \gamma'$

$$(-1)^{\alpha} z_2^{\alpha+\gamma} \left(\frac{z-z_3}{z-z_2}\right)^{\gamma} \left(\frac{z-z_1}{z-z_2}\right)^{\alpha} F\left(\alpha+\beta+\gamma,\ \alpha+\gamma+\beta';\ 1+\gamma-\gamma';\ \frac{(z-z_3)(z_1-z_2)}{(z-z_2)(z_1-z_3)}\right)$$

let $z_1 = 0$, $z_2 = \infty$, $z_3 = 1$ we get

$$z^{\alpha}(-z+1)^{\gamma} F(\alpha+\beta+\gamma,\ \alpha+\beta'+\gamma;\ 1+\gamma-\gamma';\ 1-z)$$

change $\gamma \to \gamma'$

$$z^{\alpha}(1-z)^{\gamma'} F(\alpha+\beta+\gamma',\ \alpha+\beta'+\gamma';\ 1+\gamma'-\gamma;\ 1-z)$$

$\cancel{g_2 = z^{\alpha}(1-z)^{\gamma} F(\alpha+\beta+\gamma,\ \alpha+\beta'+\gamma;\ 1+\gamma-\gamma';\ 1-z)}$

$\cancel{g_3} =$

for our case $\alpha = 0$, $\alpha' = 1-c$, $\beta = a$, $\beta' = b$
$\gamma = 0$, $\gamma' = c-a-b$

about $z=1$ g_2 g_3

$$u(z) = \alpha F(a, b; a+b+1-c; 1-z) + \beta (1-z)^{c-a-b} F(c-b, c-a; 1+c-a-b; 1-z)$$

$u(z) = z^r \, w\left(\frac{1}{z}\right)$

$z(1-z)u'' + [c-(1+a+b)z]u' - ab\,u = 0$

$\Rightarrow z(1-z)w'' + [1-a-b-2r-(2-c-2r)z]w'$
$+ [r^2 - r + rc - \frac{1}{z}(r+a)(r+b)]w = 0$

$r = -a, \quad r = -b$

$$u(z) = A\, z^{-a}\, \underbrace{F\left(a,\, 1+a-c;\, a-b+1;\, \frac{1}{z}\right)}_{g_4} + B\, z^{-b}\, \underbrace{F\left(b,\, 1+b-c;\, b-a+1;\, \frac{1}{z}\right)}_{g_5}$$

Some special cases

(i) $F(a,b,b;z) = \sum \frac{\Gamma(a+n)}{\Gamma(a)\Gamma(n+1)} z^n = (1-z)^{-a}$

(ii) $F\left(\frac{1}{2},\frac{1}{2},\frac{3}{2},z^2\right) = \frac{1}{z}\sin^{-1} z$ $\qquad \left(z\, u(z^2) = \sin^{-1} z \right)$

(iii) $F(1,1;2;-z) = \frac{\ln(1+z)}{z}$ (already)

Jacobi Functions

$$(1-z^2)u'' + [b-a-(a+b+2)z]u' + \lambda(\lambda+a+b+1)u = 0$$

$$P_\lambda^{a,b}$$

argument

$z = 1-2z'$ $\qquad (1-z)(1+z) = (1-1+2z')(1+1-2z')$

$dz = -2dz'$ $\qquad = 4z'(1-z')$

$u' = u_{z'}\left(-\frac{1}{2}\right)$

$u'' = \frac{1}{4} u_{z'z'}$

$$z'(1-z')u_{z'z'} - \frac{1}{2}[b-a-(a+b+2)(1-2z')]u_{z'} + \lambda(\lambda+a+b+1)u =$$

$$z'(1-z')u_{z'z'} - [a+1 - (a+b+2)z']u_{z'} + \lambda(\lambda+a+b+1)u = 0$$

$F\left(-\lambda, \lambda+a+b+1, a+1, \frac{1-z}{2}\right)$ Hypergeomet fn

$$P_\lambda^{(a,b)} = \frac{\Gamma(\lambda+a+1)}{\Gamma(\lambda+1)\Gamma(a+1)} F\left(-\lambda, \lambda+a+b+1, a+1, \frac{1-z}{2}\right)$$

SET 7

MATH 543: FUCHSIAN DIFFERENTIAL EQUATIONS HYPERGEOMETRIC FUNCTION

References: DK and Sadri Hassan.

Historical Notes: Please read the book *Linear Differential Equations and the Group Theory* by *Jeremy J. Gray , Birkhouser, 2000* for the contributions of **Euler, Pfaff , Gauss, Riemann, Kummer, Jacobi** and others on the P symboll and on the Hypergeometric equation and hyoergeometric function.

Definition 1. Linear ordinary differential equations having only regular singular points are called *Fuchsian Differential Equations* (FDE).

FDE with two regular singular points

Definition 2. (Regular singular point at ∞). If the transformed DE by $z = \frac{1}{t}$ has regular singular point at $t = 0$ then the original DE has a regular singular point (in the extended complex plane) at $z = \infty$.

Proposition 1. *Second order FDE having only two singular points is equivalent to a DE with constant coefficients, hence solvable in terms of the elementary functions* $\sin z$,$\cos z$ *and polynomial in z.*

FDE with three regular singular points: The Riemann equation and Riemann P Symbol.

Riemann has put a second order linear FDE with three regular singular points $z = z_1$, $z = z_2$ and $z = z_3$ into the form

$$u'' + \left(\frac{1+\alpha+\alpha'}{z-z_1}\frac{1+\beta+\beta'}{z-z_2} + \frac{1+\gamma+\gamma'}{z-z_3}\right) + \left(\frac{(z_1-z_2)(z_1-z_3)\alpha\,\alpha'}{z-z_1} + \frac{(z_2-z_1)(z_2-z_3)\beta\beta'}{z-z_2} + \frac{(z_3-z_1)(z_3-z_1)\gamma\,\gamma'}{z-z_3}\right)\frac{u}{(z-z_1)(z-z_2)(z-z_3)} = 0, \quad (1)$$

where $\alpha, \alpha', \beta, \beta', \gamma, \gamma'$ are constant satisfying the constraint

$$\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1$$

The solution of (1) is denoted as the Riemann P-symbol

$$u(z) = P \left\{ \begin{array}{cccc} z_1 & z_2 & z_3 & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{array} \right\} \tag{2}$$

The columns in the P symbol indicate the location of the regular singular points and the corresponding roots of the indical equation, i.e.,

1. $z = z_1$, $r_1 = \alpha$ and $r_2 = \alpha'$.
2. $z = z_2$, $r_1 = \beta$ and $r_2 = \beta'$.
3. $z = z_3$, $r_1 = \gamma$ and $r_2 = \gamma'$.

One can reduce the 9 number of parameters in the Riemann equation (or in the Riemann P symbol) into three parameters by using the following type of transformation

(i) $u(z) = (z - z_1)^r \, (z - z_2)^s \, (z - z_3)^t \, v(z), \quad r + s + t = 0$
(ii) The Mobius transformation: $z' = \frac{(z-z_1)\,(z_3-z_2)}{(z-z_2)\,(z_3-z_1)}$

we obtain first (from (i) above)

$$P \left\{ \begin{array}{cccc} z_1 & z_2 & z_3 & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{array} \right\} = \left(\frac{z - z_1}{z - z_2}\right)^{\alpha} \left(\frac{z - z_3}{z - z_2}\right)^{\gamma} P \left\{ \begin{array}{cccc} z_1 & z_2 & z_3 & \\ 0 & a & 0 & z \\ 1 - c & b & c - a - b & \end{array} \right\} \tag{3}$$

and the using (ii) we obtain

$$P \left\{ \begin{array}{cccc} z_1 & z_2 & z_3 & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{array} \right\} = \left(\frac{z - z_1}{z - z_2}\right)^{\alpha} \left(\frac{z - z_3}{z - z_2}\right)^{\gamma} P \left\{ \begin{array}{cccc} 0 & \infty & 1 & \\ 0 & a & 0 & z' \\ 1 - c & b & c - a - b & \end{array} \right\}, \tag{4}$$

where

$$
\begin{aligned}
z' &= \frac{(z-z_1)(z_3-z_2)}{(z-z_2)(z_3-z_1)}, \quad a = \alpha+\beta+\gamma, && (5)\\
b &= \alpha+\beta'+\gamma, \quad c = 1+\alpha-\alpha' && (6)
\end{aligned}
$$

The P symboll in the right hand side of (4) is the hypergeomtric function, $F(a,b;c;z')$

$$
F(a,b;c;z) = P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & a & 0 & z \\ 1-c & b & c-a-b & \end{matrix}\right\} \tag{7}
$$

Problems

1. Prove that the hypergeometric function defined above (7) satisfies the differential equation (hypergeometric DE)

$$
z(1-z)\,u'' + [c-(1+a+b)\,z)\,u' - ab\,u = 0 \tag{8}
$$

2. Find (Kummer) transformations leaving the Reiemann equation form invariant.

3. Prove that

$$
F = \sum_{n=0}^{\infty} \frac{a_n\, b_n}{c_n}\frac{z^n}{n!}
$$

where

$$
a_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad b_n = \frac{\Gamma(b+n)}{\Gamma(b)}, \quad c_n = \frac{\Gamma(c+n)}{\Gamma(c)} \quad |z| \leq 1
$$

4. Prove the following Proposition.

Proposition 2. *Solutions of the hypergeometric function bout its regular singular points $z=0$, $z=\infty$ and $z=1$, provided $1-c$, $b-a$, and $c-a-b$ are not integers, are respectively given by*

$$
\begin{aligned}
u(z) &= A_1\, F(a,b;c;z) + B_1\, z^{1-c}\, F(b-c+1, a-c+1; 2-c; z), && (9)\\
u(z) &= A_2\, z^{-a}\, F(a, a-c+1; a-b+1; \frac{1}{z}) + B_2\, z^{-b} \\
&\quad F(b, b-c+1; b-a+1; \frac{1}{z}), && (10)\\
u(z) &= A_3\, F(a,b; a+b+1-c; 1-z) + B_3\, z^{c-a-b}\, F(c-b, c-a; \\
&\quad 1+c-a-b; 1-z), && (11)
\end{aligned}
$$

where $A_i, B_i, (i = 1,2,3)$ *are arbitrary constants.*

5. Find the solutions (10) and (11) about the regular singular points $z = \infty$ and $z = 1$ by the use of *Kummer's* transformations mentioned in the Pr.2.

6. Prove the following: The Jacobi function $P_\lambda^{(\alpha,\beta)}$ satisfying the Jacobi equation equation

$$(1-z^2)u'' + [\beta - \alpha - (\alpha+\beta+2)z]u' + \lambda(\lambda+\alpha+\beta+1)u = 0$$

can be written in terms of the hypergeometric function

$$P_\lambda^{(\alpha,\beta)} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}\, F(-\lambda, \lambda+\alpha+\beta; \alpha+1; \frac{1-z}{2})$$

When $\lambda = n$ a non-negative integer the the solution becomes Jacobi polynomials containing the Legendre, and Tchbechev polynomials.

7. Prove that when the regular singular points $z = z_2$ and z_3 of the Riemann equation are pushed out ∞ then the resulting function function is the confluent hypergeometric function $\Phi(a,c;z)$ in

8. Prove that $\Phi(a,c,z) = \lim_{b\to\infty} F(a,b;c;\frac{z}{b})$

9. Prove that the confluent hypergeometric function satisfies the DE

$$z\,u'' + (c-z)\,u' - a\,u = 0 \tag{12}$$

10. Prove that the point $z = \infty$ of the confluent hypergeometric equation (12) is an irregular singular point.

11. Find solutions of the confluent hypergeometric equation about all it regular singular points. The form of the equation is of the form

$$u(z) = A\,\Phi(a, c; z) + B\, z^{1-c}\,\Phi(a', c', b'\, z)$$

where A and B are arbitrary constants. Find the constants a', c' and b'

12. Prove that Bessel's function $J_v(z)$ satisfying the Bessel equation $u'' + \frac{1}{z}u' + (1 - \frac{\nu^2}{z^2})u = 0$ is given by

$$J_v(z) = \frac{1}{\Gamma(\nu+1)}(z/2)^\nu\,\Phi(\nu + 1/2, 2\nu + 1; 2iz)$$

13. Prove that

$$\frac{d^n}{dz^n}F(a, b; , c; z) = \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+n)}F(a+n, b+n; c+n; z)$$

Use induction.

14. Prove the following:

(i) $F(-a, b; b; -z) = (1+z)^a$,

(ii) $F(1, 1; 2; -z) = \frac{1}{z}\ln(1+z)$,

(iii) $F(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2) = \frac{1}{z}\sin^{-1} z$,

(iv)

$$F(\frac{1}{2}, \frac{1}{2}; 1; z^2) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - z^2\sin^2\theta}},$$

(v) $e^z = \lim_{b\to\infty} F((1, b, 1, \frac{z}{b})$

(vi) The error function

$$Erf(z) = \int_0^\infty e^{-t^2}\,dt$$

15. Prove that the Hermite-Weber differential equation

$$u'' + (\nu + \frac{1}{2} - \frac{1}{4}z^2)u = 0$$

can be converted to the confluent hypergeometric equation by

$$u(z) = e^{\frac{-z^2}{4}}\,v(\xi), \quad \xi = \frac{z^2}{2}$$

with

$$v(\xi) = \Phi(-\nu/2, 1/2; \xi)$$